Post 5: Pivotal and Ribbon Categories

In this fourth post I will finish Chapter 3 of Chris Heunen and Jamie Vicary's Categories for Quantum Theory. Here I will introduce the series of additional types of categories stemming from the presence of duals. This includes pivotal categories which are equipped with monoidal natural transformations from objects to their double duals, i.e. have an identification between taking the dual of the dual, and doing nothing. Another important structure I'll introduce today is that of balanced monoidal categories which are equipped with the ability to "twist" wires, as well as ribbon categories, for which wires behave rather as ribbons in 3D space.

1 Pivotal and Balanced Categories

Definition (Pivotal category) A monoidal category with right duals is pivotal when it is equipped with a monoidal natural transformation $\pi_A : A \to A^{**}$.

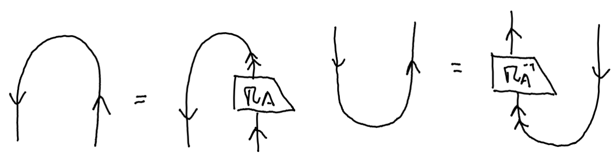
In particular, π_A must satisfy

$$\begin{array}{ccc} & A \otimes B & \\ {}^{\pi_A \otimes \pi_B} \swarrow & & \searrow {}^{\pi_{A \otimes B}} \\ A^{**} \otimes B^{**} & \xrightarrow[\phi_{A,B}]{} & (A \otimes B)^{**} \end{array} \qquad\qquad \pi_I = \psi$$

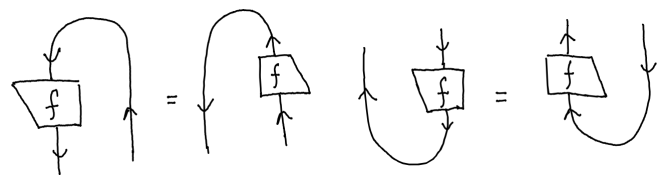
where $\phi_{A,B}$ and $\psi : I \to I^{**}$ are the canonical isomorphisms arising from the double dual lemma.

Lemma (Invertibility of π_A) In a pivotal category, the morphisms $\pi_A : A \to A^{**}$ are invertible.

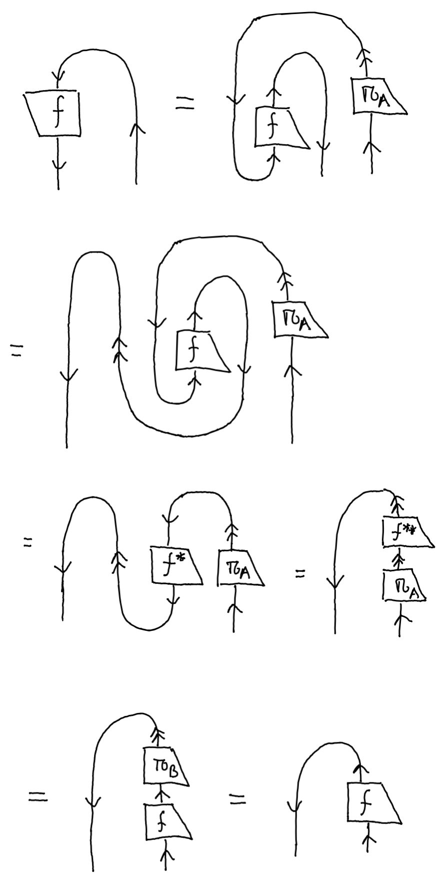
Observation In a pivotal category, one has additional cups and caps



Lemma (Sliding) In a pivotal category, for all morphisms $f : A \to B$:



Proof The proof of the sliding theorem proceeds graphically as

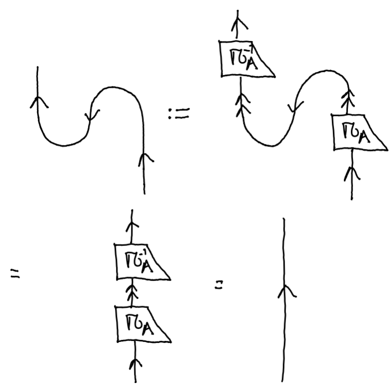


The other equality is proved analogously.

Observation A pivotal structure essentially says that taking right duals twice is equivalent to doing nothing. However, since taking left and right duals are inverse processes, then a pivotal structure can also be interpreted as an equivalence between left duals and right duals.

Theorem (Left duals exist in a pivotal category) In a pivotal category, every object has a left dual.

Proof Due to the existence of the morphism π_A, we have



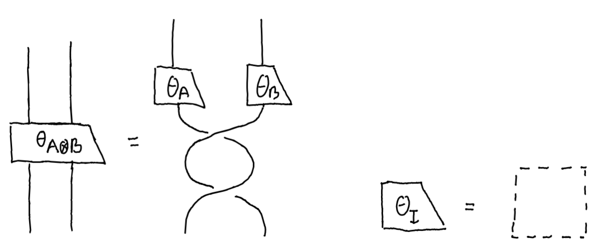
The other axiom for left duality is proved similarly.

Theorem (Correctness of the graphical calculus for a pivotal category) In a pivotal category, a well-formed equation between morphisms follows from the axioms if and only if it holds in the graphical language up to planar oriented isotopy.

The key new feature of this correctness theorem is the word oriented, i.e. a valid isotopy must preserve the arrows in the wires of any diagram.

Observation In the presence of a braiding, pivotal structure can be expressed in terms of a twist. This pushes us to the concept of a balanced monoidal category.

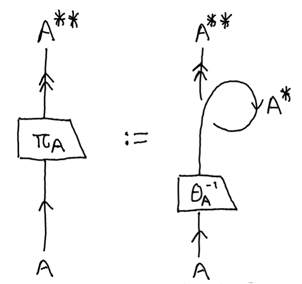
Definition (Balanced monoidal category) A braided monoidal category is called balanced when it is equipped with a natural isomorphism $\theta_A : A \to A$ called a twist, satisfying the equations



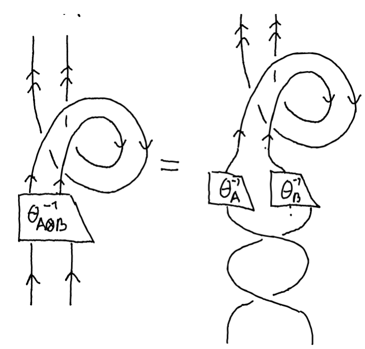
Observation Every symmetric monoidal category admits the trivial twist $\theta_A = \mathrm{id}_A$.

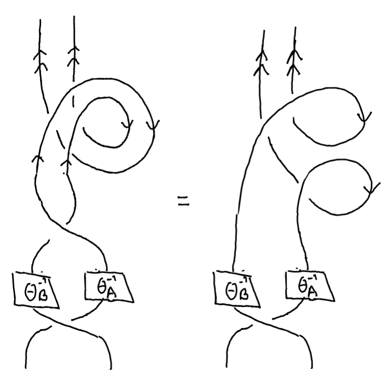
Theorem (Equivalence of pivotality and balance in braided monoidal categories) In a braided monoidal category with right duals, a pivotal structure uniquely induces a twist structure, and vice-versa.

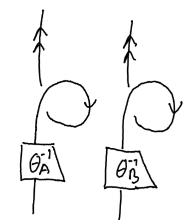
Proof Givevn right duals for every object A, and a twist structure θ_A, we can define the following pivotal structure π_A



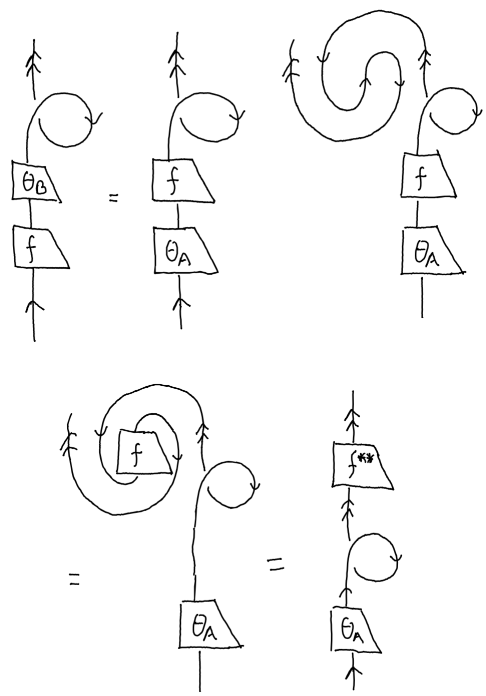
π_A is by construction a morphism from A to A^{**}. To show that it is a pivotal structure, it then suffices to show that it is natural and monoidal. Let's first show that it is a monoidal transformation



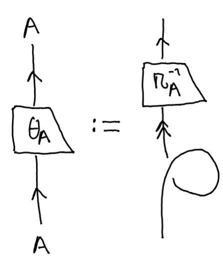




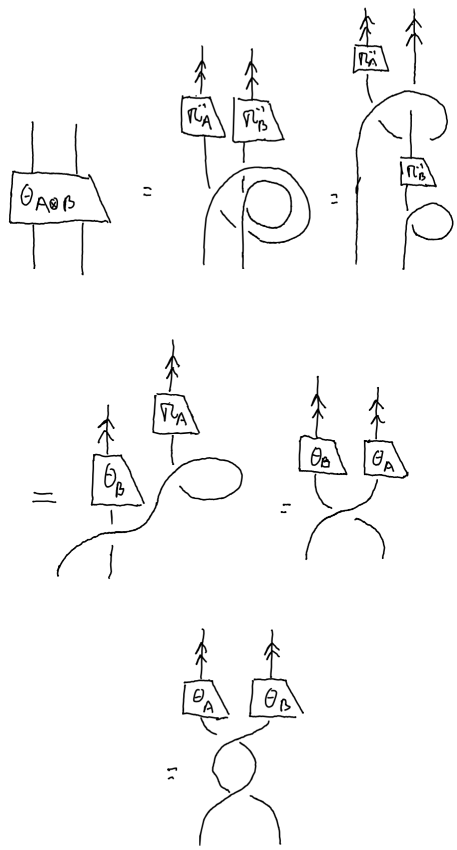
Now let's show that it is a natural transformation



Conversely, given a pivotal structure π_A, a balanced structure, or a twist, can be defined as



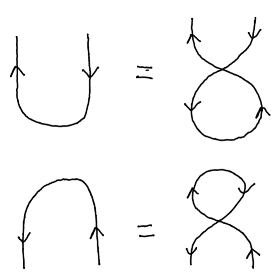
We then have to prove the balanced equation, which we can do by expanding

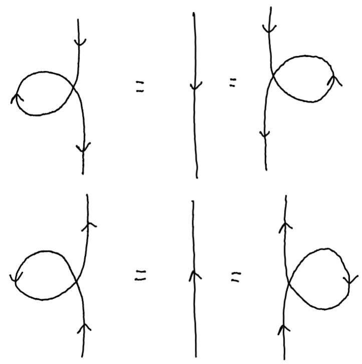


Definition (Compact category) A compact category is a pivotal symmetric monoidal category where the canonical twist is the identity $\theta_A = \mathrm{id}_A$.

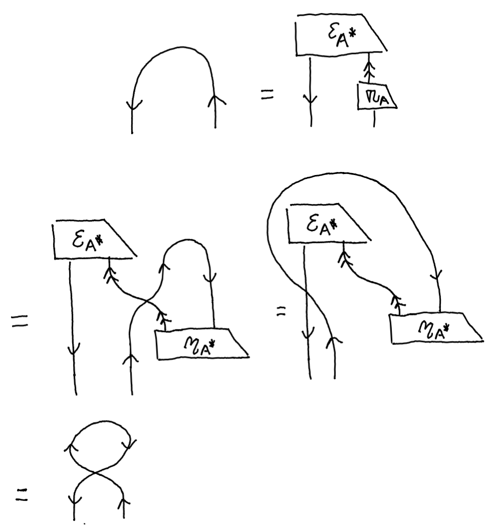
Observation Any symmetric monoidal category in which every object has a right dual is compact in a canonical way.

Lemma In a compact category, the following equations hold





Proof Let us prove some the second equality in a diagrammatic fashion. We have



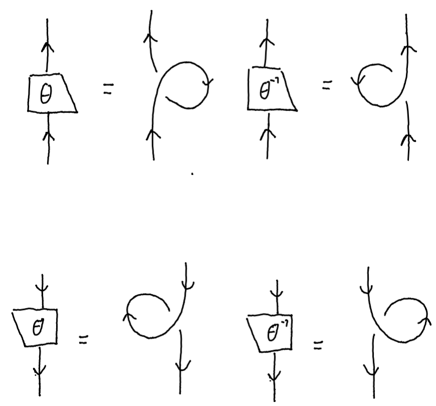
Proofs for the other equalities can be similarly obtained.

2 Ribbon Categories

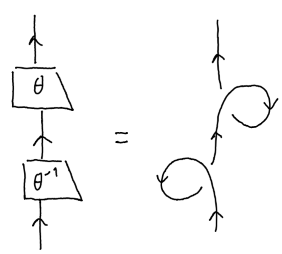
When using the graphical calculus for braided pivotal categories, one must be careful with loops on a single strand. Indeed, the correctness of the graphical calculus does not imply that a loop must equal the identity, and this is because the correctness theorem only allows for planar oriented isotopy, not spatial oriented isotopy.

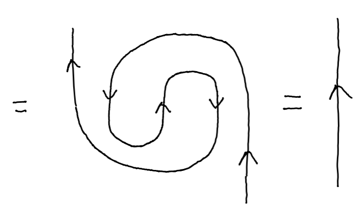
Therefore loops cannot be generically removed. In fact, a loop on a single strand is related to the twist.

Lemma (Loop = Twist) In a braided pivotal category, the following equations hold

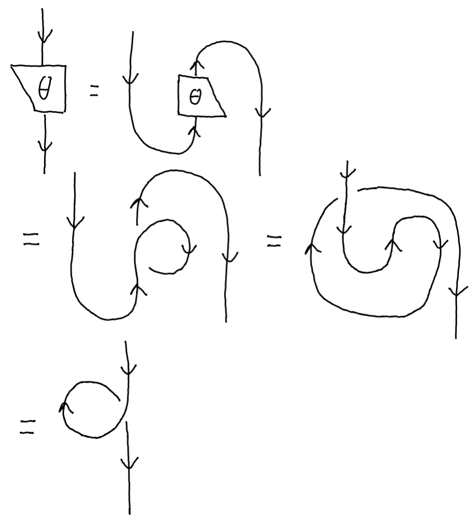


Proof The first equality stems directly from the definition of the twist in terms of the pivotal structure. The equation for θ^{-1} can be verified by computing



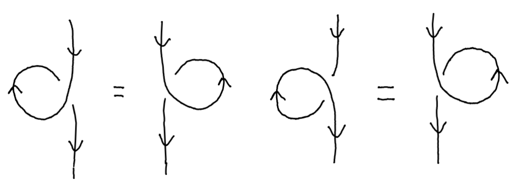


Due to the uniqueness of inverses in a category, the expression for θ must be correct. As for the graphical form of θ^*, we have



Definition (Ribbon category) A ribbon category is a balanced monoidal category with duals, such that $(\theta_A)^* = \theta_{A^*}$.

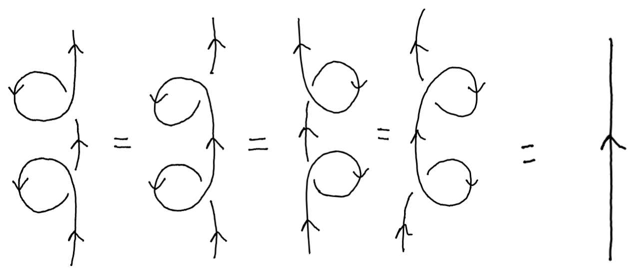
Lemma A balanced monoidal category is a ribbon category if and only if either of the following equations are satisfied



Proof This follows directly from the definition of the twist in a braided pivotal category as given above.

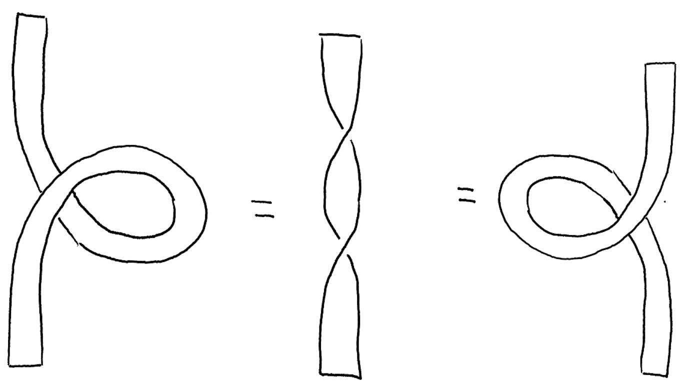
Lemma A compact category is a ribbon category.

Lemma In a ribbon category, the following equations hold



Proof All of these equalities are consequences of the definitions of θ, θ^*, θ^{-1} and $\left(\theta^{-1}\right)^*$.

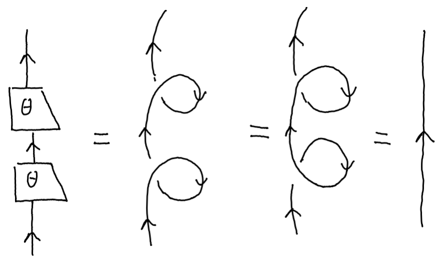
Observation These are the equations we would expect to be satisfied by ribbons in an ambient 3D space. This is made precise by the correctness theorem for ribbon categories.



Theorem (Correctness of the graphical calculus for ribbon categories) A well formed equation between morphisms in a ribbon category follows from the axioms if and only if it holds in the graphical language up to framed isotopy in 3D. Framed isotopy means that strands are thought of as ribbons rather than wires.

Lemma In a symmetric ribbon category $\theta \circ \theta = \mathrm{id}$.

Proof Graphically, we have

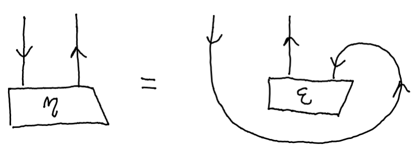


3 Dagger duality

Lemma In a monoidal dagger category $L \dashv R \Leftrightarrow R \dashv L$.

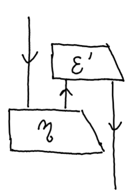
Proof This follows directly from the axiom $(f \otimes g)^\dagger = f^\dagger \otimes g^\dagger$ for monoidal dagger categories.

Definition (Dagger dual) In a dagger category with a pivotal structure, a dagger dual is a duality $A \dashv A^*$ witnessed by morphisms $\eta : I \to A^* \otimes A$ and $\varepsilon : A \otimes A^* \to I$ that satisfy

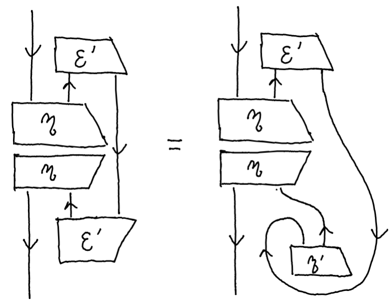


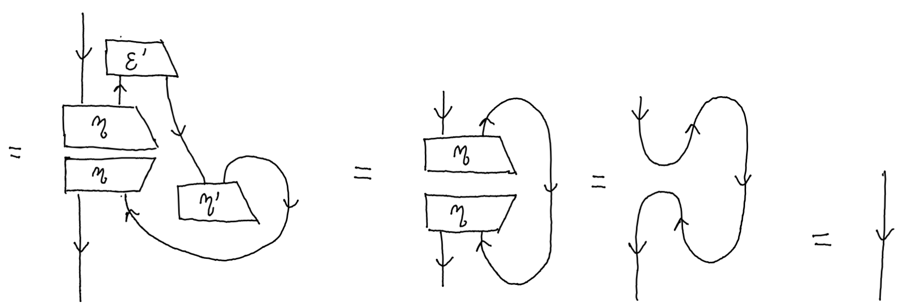
Lemma In a dagger pivotal category with a pivotal structure, dagger duals are unique up to unique unitary isomorphism.

Proof Given dagger duals $(L \dashv R, \eta, \varepsilon)$ and $(L \dashv R', \eta', \varepsilon')$, we have the isomorphism $R \simeq R'$ given by



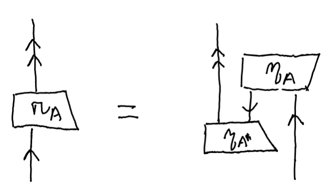
Then, we can show that this is a co-isometry as



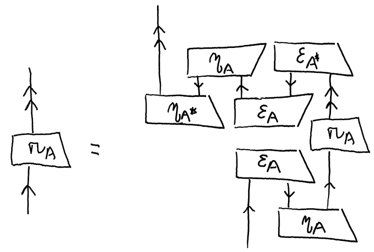


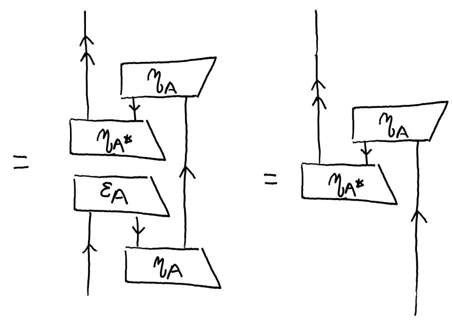
Definition (Dagger pivotal category) A dagger pivotal category is a monoidal category with a pivotal structure, such that the chosen right duals are all dagger duals.

Proposition The pivotal structure in a dagger pivotal category is given by the composite



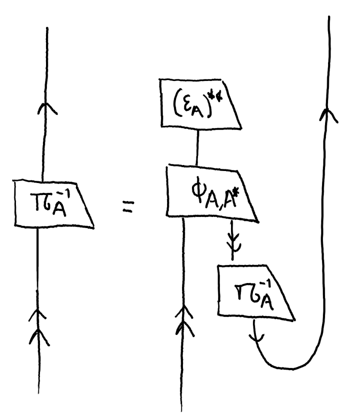
Proof The proof proceeds graphically as

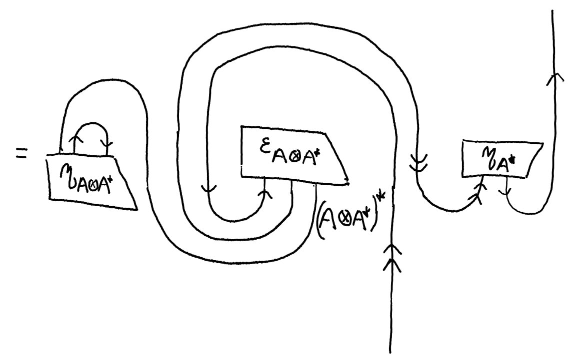


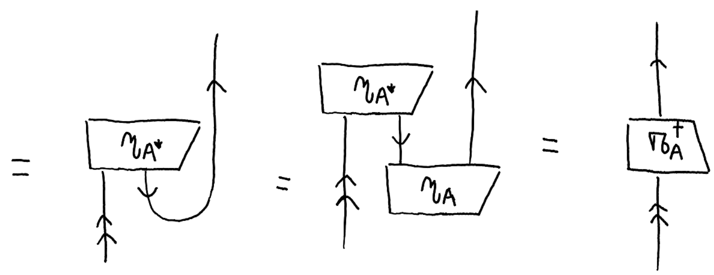


Proposition In a dagger pivotal category, the pivotal structure is unitary

Proof Using the canonical isomorphism ϕ_{A,A^*}, we can write



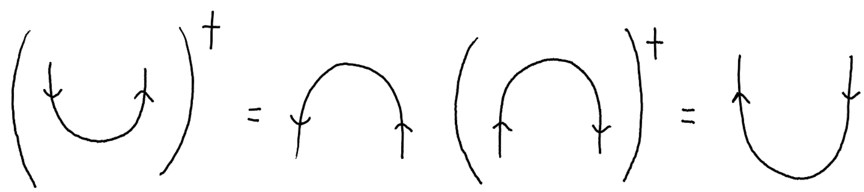




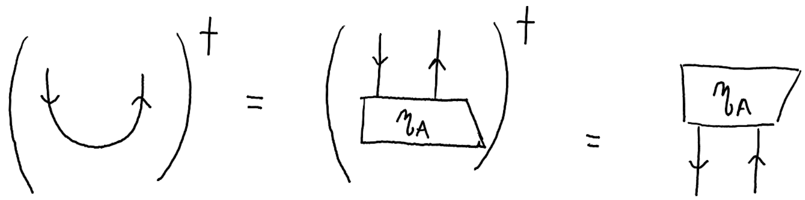
which completes the proof.

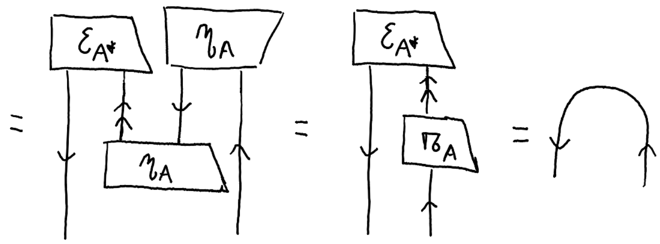
Observation Dagger pivotal categories have a graphical calculus where the dagger acts as reflection along a horizontal axis.

Lemma In a dagger pivotal category, the following equations hold



Proof We proceed graphically as

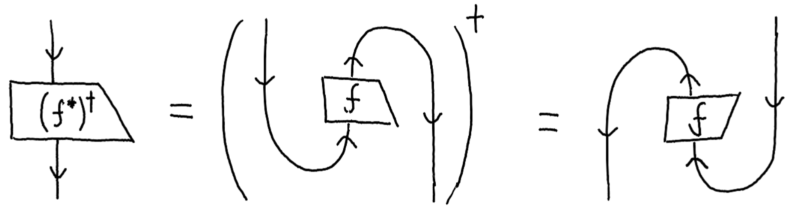


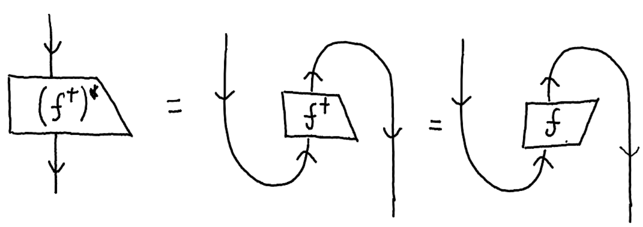


The second equation follows by the fact that the cup determines the cap, proved in the previous blog post.

Lemma (The dagger and dual functors commute) In a dagger pivotal category, every morphism f satisfies $(f^*)^\dagger = \left(f^\dagger\right)^*$.

Proof We can compute both sides



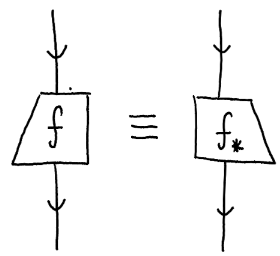


And since the results are isotopic, then we have proved the equality.

Definition (Conjugation) On a dagger pivotal category, conjugation $(\cdot)_*$ is the composite of the dagger functor and the right-dual functor

$$(\cdot)_* := (\cdot)^{*\dagger} = (\cdot)^{\dagger *}. \tag{1}$$

Observation Conjugation is denoted graphically by reflection along a vertical axis.

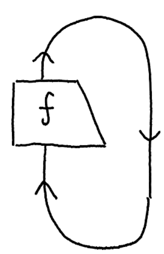


Definition (Ribbon dagger category) A ribbon dagger category is a braided dagger pivotal category with unitary braiding and twist.

Definition (Compact dagger category) A compact dagger category is a symmetric dagger pivotal category with unitary symmetry and $\theta = \mathrm{id}$.

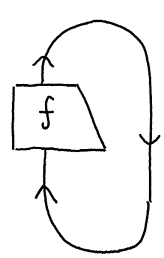
4 Traces

Definition (Trace) In a pivotal category, the trace of a morphism $f : A \to A$ is the following scalar



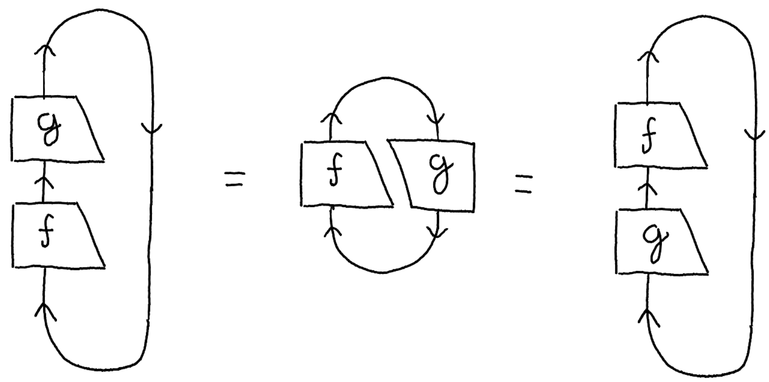
It is denoted by $\mathrm{tr}\,(f)$ or $\mathrm{tr}_A\,(f)$.

Definition (Dimension) The dimension of an object A is the scalar $\dim A = \mathrm{tr}\,(\mathrm{id}_A)$. Graphically it reads



Lemma (Cyclic property of the trace) In a pivotal category, $\mathrm{tr}_A(g \circ f) = \mathrm{tr}_B(f \circ g)$ for $f : A \to B$ and $g : B \to A$.

Proof Graphically



Lemma (Properties of the trace) In a pivotal monoidal category, the trace has the following properties

1. $\mathrm{tr}_A(f + g) = \mathrm{tr}_A(f) + \mathrm{tr}_A(g)$ for any superposition rule $+$
2. $\mathrm{tr}_{A \oplus B}\begin{pmatrix} f & g \\ h & j \end{pmatrix} = \mathrm{tr}_A(f) + \mathrm{tr}_B(j)$ if there are biproducts
3. $\mathrm{tr}_I(s) = s$ for any scalar $s : I \to I$
4. $\mathrm{tr}_A(0_{A,A}) = 0_{I,I}$ if there is a zero object
5. $\mathrm{tr}_{A \otimes B}(f \otimes g) = \mathrm{tr}_A(f) \circ \mathrm{tr}_B(g)$ in a braided pivotal category
6. $(\mathrm{tr}_A(f))^\dagger = \mathrm{tr}_A(f^\dagger)$ in a dagger pivotal category

Lemma (Properties of dimensions) In a braided pivotal category, the following properties hold

1. $\dim(A \oplus B) = \dim A + \dim B$ if there are biproducts
2. $\dim(I) = \mathrm{id}_I$
3. $\dim(0) = 0_{I,I}$ if there is a zero object
4. $A \simeq B \Rightarrow \dim A = \dim B$
5. $\dim(A \otimes B) = \dim A \circ \dim B$

5 More to come

In the next post I will move on to introducing notions necessary for reading the paper "An invitation to topological orders and category theory" by Kong and Zhang. In particular, I will focus on $\mathbb{C}-$linear categories, simple and semisimple $\mathbb{C}-$linear categories and their interaction with all other types of structures we have introduced so far. This will lead naturally into the definition of (unitary) fusion categories, and modular tensor categories right after, which set the stage for building up the physical picture of topological order, coming up in the post after the next.